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A New Geometric School Text from Cornell

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The school text CUNES 52-02-043 gives the solution to a geometric exercise which is to compute the circumferences and the areas of the annuli defined by a system of eight concentric circles. It therefore probably shows a pupil's sketchy solution of an assignment as it is known from a number of problem texts.

Figure 1: CUNES 52-02-043 obv. Photograph by Laura Johnson-Kelly.

*I would like to thank Professor David I. Owen, Curator of Tablet Collections, for bringing this tablet to my attention, encouraging its publication, and for arranging for the excellent photo by Laura Johnson-Kelly. This study was done while I held a research position at the Exzellenzcluster 264 TOPOI "The Formation and Transformation of Space and Knowledge in Ancient Civilisations."

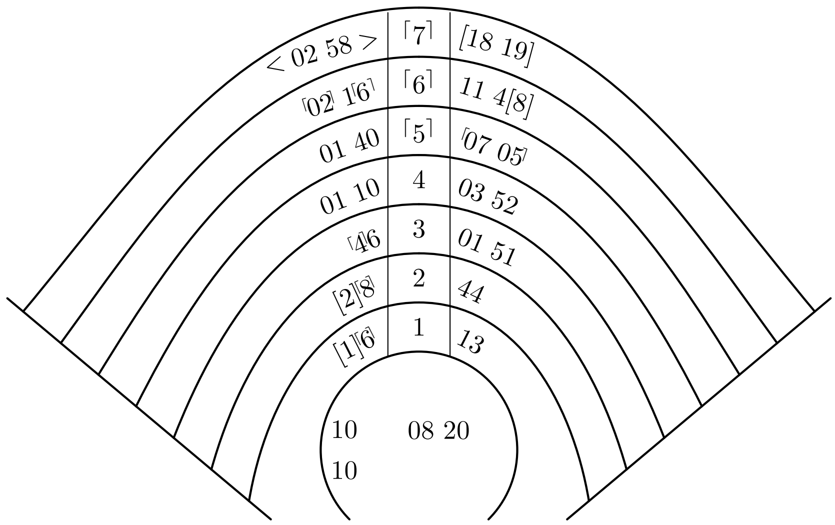

Figure 2: Schematic transliteration of the drawing on CUNES 52-02-043 obv.

The obverse of the lentil-shaped school tablet CUNES 52-02-043 (90 mm × 92 mm × 32 mm, reverse empty, provenance unknown) shows the solution to a geometric exercise dealing with a system of eight concentric circles (figs. 1 and 2). It consists of a scetchy drawing with inscribed numbers representing various length and area values. The drawing depicts only a sector-shaped section of the figure involved and is highly out of scale. In fact, though the annuli bounded by the subsequent circles increase in width by one at each step (see fig. 3), the circles are drawn equidistantly. In the following I will use the symbol $\hat{\pi}$ (with value 3) to denote the Mesopotamian equivalent to our π.[1] The numbers given are one possible interpretation. Statements remain valid when at the same time numbers representing lenghts are multiplied by 60^k, and those representing areas by 60^{2k} for arbitrary integer k.

The interpretation is the following. The innermost circle ("circle 0" in the following) has the area $A_0 = 08;20$ from which its circumference c_0 and radius r_0 result as

$$c_0 = \sqrt{4\hat{\pi}A_0} = 10 \qquad \text{and} \qquad r_0 = \frac{c_0}{2\hat{\pi}} = 01;40\,. \tag{1}$$

The circumference c_0 seems to be written twice at the left side inside the inner circle. The seven outer circles will be referred to as "circle 1, circle 2, …", in the order of increasing diameter. As indicated by the numbers in the central "column" of the drawing, the widths of the annuli bounded by the subsequent circles (i.e., the increments of the circles' radii) are $\Delta_i = i$ $(1 \leq i \leq 7)$, i.e. increase arithmetically from inside to outside,

[1]Purists and those afraid of "anachronisms" won't like that but it is practical and does no harm.

whence radius r_n and circumference c_n of circle n $(1 \leq n \leq 7)$ are[2]

$$r_n = r_0 + \sum_{i=1}^{n} \Delta_i = r_0 + \sum_{i=1}^{n} i = 01;40 + \frac{1}{2}n(n+1) \tag{2}$$

and

$$c_n = 2\hat{\pi} r_n = 6 r_n = 6\left(01;40 + \frac{1}{2}n(n+1)\right) = 10 + 3n(n+1)\,. \tag{3}$$

The area $\tilde{A}_n$ of the n-th annulus (bounded by the n-th and $(n-1)$-th circle) can either be found by computing the areas A_n and A_{n-1} of the bounding circles as

$$A_n = \frac{1}{4\hat{\pi}} c_n^2 \qquad \text{and} \qquad A_{n-1} = \frac{1}{4\hat{\pi}} c_{n-1}^2$$

and then $\tilde{A}_n = A_n - A_{n-1}$, or directly from the circumferences c_n and c_{n-1} as[3]

$$\tilde{A}_n = \Delta_n \frac{c_n + c_{n-1}}{2} = n\,\frac{c_n + c_{n-1}}{2}\,. \tag{4}$$

The numerical values for the circumferences c_n of the circles, the radial increments Δ_i, and the areas $\tilde{A}_n$ of the annuli are collected in the following table. These are the respective values written in the left, central, and right column of the drawing. The situation is depicted to scale in fig. 3.

$c_1 = 16$	$\Delta_1 = 01$	$\tilde{A}_1 = 13$
$c_2 = 28$	$\Delta_2 = 02$	$\tilde{A}_2 = 44$
$c_3 = 46$	$\Delta_3 = 03$	$\tilde{A}_3 = 01\ 51$
$c_4 = 01\ 10$	$\Delta_4 = 04$	$\tilde{A}_4 = 03\ 52$
$c_5 = 01\ 40$	$\Delta_5 = 05$	$\tilde{A}_5 = 07\ 05$
$c_6 = 02\ 16$	$\Delta_6 = 06$	$\tilde{A}_6 = 11\ 48$
$c_7 = 02\ 58$	$\Delta_7 = 07$	$\tilde{A}_7 = 18\ 19$

[2]Note that the equality $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ was known and made explicit use of by the Babylonian mathematicians. See Friberg (1987–90, 578); Robson (1999, 80-81).

[3]This follows by

$$\begin{aligned}\tilde{A}_n &= A_n - A_{n-1} = \frac{1}{4\hat{\pi}}(c_n^2 - c_{n-1}^2) = \frac{1}{4\hat{\pi}}(c_n - c_{n-1})(c_n + c_{n-1}) \\ &= \frac{1}{4\hat{\pi}} \cdot 2\hat{\pi}(r_n - r_{n-1})(c_n + c_{n-1}) = \Delta_n \frac{c_n + c_{n-1}}{2} = n\,\frac{c_n + c_{n-1}}{2}\,.\end{aligned}$$

Because of $c_n = 10 + 3n(n+1)$ the $\tilde{A}_n$ satisfy

$$\tilde{A}_n = \frac{n}{2}\left(10 + 3n(n+1) + 10 + 3(n-1)n\right) = \frac{n}{2}(20 + 6n^2) = 10n + 3n^3\,.$$

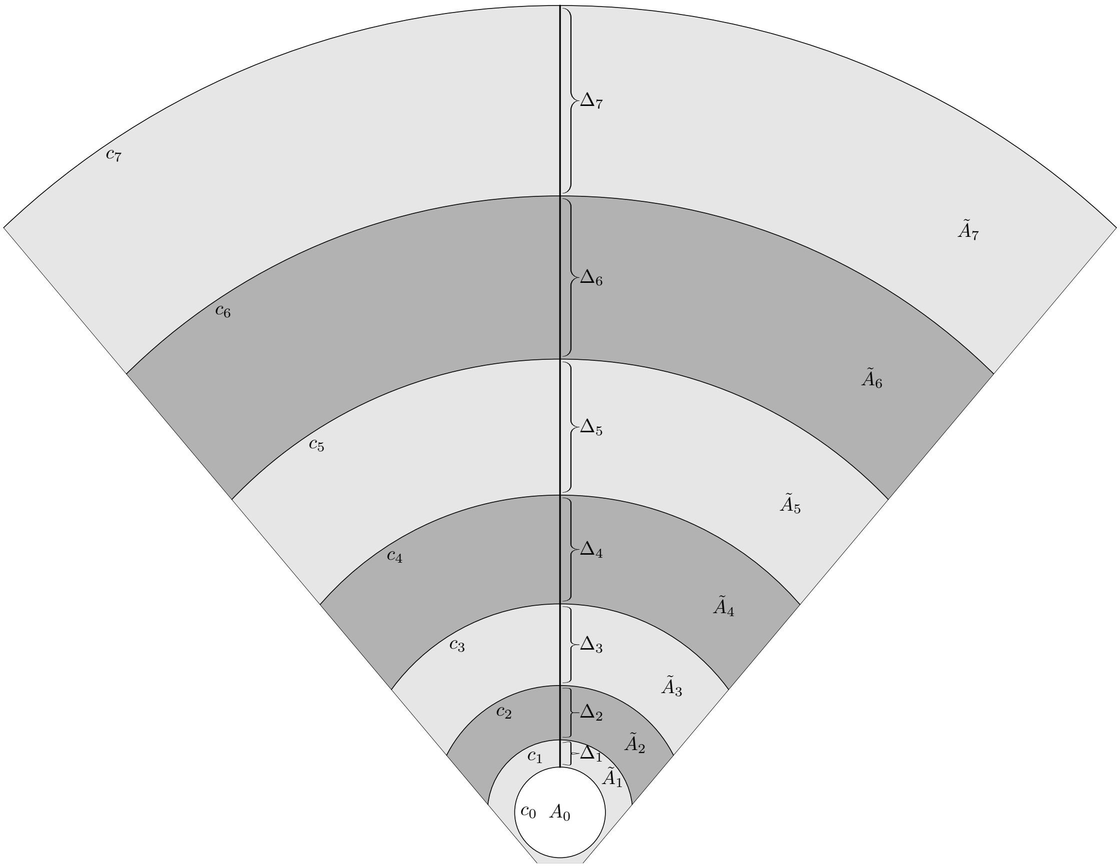

Figure 3: The situation of CUNES 52-02-043 obv. drawn to scale

The underlying problem text would most likely have given the central circle's area 08 20 and the arithmetically increasing sequence of radius increments $(\Delta_i)_{1\leq i\leq 7}$ and asked for the outer circles' circumferences and the areas of the annuli.[4] The solution would then start by computing the innermost circle's circumference and radius (see (1)) and proceed by computing the radii of the outer circles (either by successive addition of the radius increments Δ_i or by means of (2)) and their circumferences (3). Finally, the areas of the annuli are determined, probably by means of (4) as it is done in W 23291-x, § 2 (Friberg u. a., 1990, 494-496), see below.

There are three problems (all accompanied by drawings) that are sort of "complementary" to this one, in that given and asked data are partially exchanged:

1. W 23291-x, § 2 (Friberg u. a., 1990, 494-496): Five concentric circles. The problem gives the circumference of the *outer* circle and a *constant* radius *de*crement for the

[4]The wording would have started with something like "08 20: a circle I have bent" (**gur₂** *akpup*). See e.g. Böhl 1821 (Leemans, 1951); BM 85194, § 4 (Neugebauer, 1935, 144, 153, 167-172): *ak-pu-up*. W 23291-x, §§ 2 and 3 (Friberg, Hunger und al-Rawi, 1990, 494-496; 499-502): **gur₂**^{up}.

four inner circles (both given as well in the statement of the problem as in the drawing), and asks for the areas of the annuli and the inner circle. As established by Friberg u. a. (1990, 495), first the diameters and the circumferences of the inner circles are computed (both denoted only in the drawing) and from those the lengths of the “middle arcs” of the annuli as mean values of subsequent circumferences (given in the text, but not in the drawing). From those and the (constant) radius decrement the areas of the annuli (given in both text and drawing) are computed by means of (4).

2. W 23291-x, § 3 (Friberg u. a., 1990, 499-502): Five concentric circles. Given are the *diameter* of the inner circle and the widths of the annuli. Computed are their areas.

3. Böhl 1821 (Leemans, 1951): Two concentric circles. The problem gives the width Δ and the *area* $\tilde{A}$ *of the annulus* (which in the drawing is written inside the inner circle) and asks for the dimensions of the two circles. First the area of the annulus is divided by $\hat{\pi}\Delta$ to obtain $\frac{\tilde{A}}{\hat{\pi}\Delta} = \frac{\tilde{A}}{\hat{\pi}(r_2-r_1)} = r_2 + r_1$. Then the radius increment Δ is first added to this $r_2 + r_1$ to obtain $r_2 + r_1 + r_2 - r_1 = 2r_2 = d_2$ and then subtracted from it to obtain $r_2+r_1-(r_2-r_1) = 2r_1 = d_1$ with d_1, d_2 the respective circles’ diameters. From this the circles’ areas are computed.

There are some more signs on the tablet below the drawing which may or may not be connected to the problem of the concentric circles. On the right side immediately below the drawing there is clearly discernible 41 20 next to damaged surface. The area A_6 of the sixth circle (not annulus) is 25 41;20, but this need not mean anything.

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